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1972 J. Phys. A: Gen. Phys. 5 1320

(http://iopscience.iop.org/0022-3689/5/9/006)

J. Phys. A: Gen. Phys., Vol. 5, September 1972. Printed in Great Britain. © 1972.

Straggling of moderately relativistic electrons

J B EHRMAN

Department of Applied Mathematics, University of Western Ontario, London, Ontario, Canada

MS received 17 January 1972, in revised form 26 April 1972

Abstract. A simple analytical model for computing range-energy straggling of moderately relativistic electrons is set up. Both inelastic scattering and bremsstrahlung are considered as sources of straggling. The model gives overestimates for radiative straggling, but it is indicated how this overestimation may be removed. The simple (overestimating) model is used to compute range straggling for electrons up to kinetic energies of 4 MeV in uranium and up to 30 MeV in aluminium. The occurrence of the infinities in Eyges' radiative straggling treatment are explained and removed. The results justify the neglect of straggling in electron transport when applied to photon production, as done by Brown, Ogilvie, Wittry, Kyser. Ehrman, dePackh, and others.

1. Introduction

This is the first of two papers dealing with the range of validity of the description of electron transport in matter by a single partial differential equation for the electron flux. This study was motivated by interest in the photon spectrum emerging from a layer of material bombarded by electrons of moderately relativistic energies. Since in the intended applications, a substantial photon flux emerging from the layer (in the forward direction, ie from the face which is not bombarded by the primary electrons) is desirable, the layer thickness considered was large compared to the transport mean free path λ of the primary electrons, but small compared to the electron range. That is, the layer is of intermediate thickness, of the order of magnitude of a primary electron random walk radius in a Fermi-age theory for electrons (Bethe *et al* 1938, Meister 1958), which is about the geometric mean of transport mean free path and range.

Numerous approaches to the electron transport problem already exist. There are analytical methods, such as the various multiple scattering theories (see Zerby and Keller 1967 for a complete list of references) but these are of limited applicability at 'intermediate' thicknesses, where path lengths are larger than λ, but small enough so that boundary conditions are important. More appropriate at intermediate thicknesses are the Fermi-age theory or diffusion theory of Bethe *et al* (1938) and of Meister (1958), and the more refined Bethe *et al* (1938) transport equation which has been used by Brown (1965), Brown and Ogilvie (1966), Brown *et al* (1969) and Ehrman (1969 *US Naval Research Laboratory, Radiation Project Progress Report* no 16 unpublished). (There has been a good deal of confusion in the literature between the approximation involved in the diffusion or age equation of § 4 of Bethe *et al* (1938) and the transport equation of § 2 of Bethe *et al* (1938). Only the transport equation of § 2 will be referred to as the BRS equation here. The approximations involved are quite different, and the differences are explained in detail in the second paper of this series.) There is also the moment method of Spencer (1955, 1959), which treats the scattering of electrons more accurately than the BRS equation, but which becomes clumsy when applied with finite boundaries, rather than to an infinite medium. Finally, there are numerical methods such as Monte Carlo methods, invariant embedding, and other complex geometry methods. (In addition to the paper by Birkhoff (1958), the reader is referred to the more recent review paper of Zerby and Keller (1967) for an overall view of the theoretical and experimental situation.)

The age theory, the BRS transport equation, and the Spencer moment method all involve a continuous slowing down model for electrons, that is, it is assumed that the kinetic energy of an electron is a function only of its primary energy and of the path length traversed since entering the material. The Monte Carlo and the complex geometry methods do not make this assumption of negligible straggling, and can therefore be used where this assumption breaks down. It breaks down in inhomogeneous material, where not only the path length but the whole path history is important; it also breaks down in homogeneous material at very deep penetrations, such as shielding problems, where the spectrum is composed predominantly of 'atypical' particles; and it breaks down at highly relativistic energies where the bremsstrahlung radiative straggling becomes important.

The object of this paper is to show that for moderately relativistic electrons ($\leqslant 4$ MeV in uranium, $\leqslant 30$ MeV in aluminium), the straggling due to inelastic scattering and due to bremsstrahlung is not a large effect, and to estimate it quantitatively. Since the final use to be made of these results is for photon production, and since it has been shown by Spencer and Fano (1954) that secondary electrons are unimportant except for the very soft parts of the electron spectrum (10% correction if energy = 16% of primary energy; factor of 2 correction if energy = 4% of primary energy) which contribute negligibly to photon production, the model presented here does not take secondary electrons produced by inelastic scattering into account. We also do not consider the secondary electrons produced by photons, that is, pair production electrons, Compton electrons, and photoelectrons.

Unlike the work of Eyges (1949, 1950) and of Blunck and Westphal (1951), the present treatment of radiative straggling is not limited to small penetration, that is, to path lengths small compared to the average range; nor is any attempt made to expand functions in powers of the path length x, since $x = 0$ turns out to be a branch point for the various moments of the probability distribution. This attempt to expand in Taylor series about what is really a branch point accounts for the infinite term encountered in Eyges (1949).

Unlike Spencer and Fano (1954), we consider correlations between energy and path length, so as to determine range straggling, rather than only the energy flux spectrum of electrons as they slow down. But we use much rougher approximations for the various cross sections involved, so that the overall results can be obtained in a reasonably simple analytical form. Our method can be applied to more accurate cross sections, and the results worked out numerically, but for an estimate of the error of the continuous slowing down model applied to photon production cited above, an analytical formula which, if anything, somewhat overestimates the radiative straggling, is adequate.

We thus derive an estimate for the error in range (and photon production) for electrons in (infinite) materials of various Z for a wide range of primary kinetic energies. This defines the range of validity of the continuous slowing down model as used in the age theory, the BRS transport equation, and the Spencer moment method, provided that the material is homogeneous, and the penetration is not so deep that only atypical electrons are important and a few moments of distributions are unimportant. By this, we mean that while *path lengths* comparable to the average range are properly treated, path lengths larger than the average range are not. The *thickness* of the layer considered must therefore be somewhat less than the average path length, but of the order of the age-theory random walk radius or a few times that, if it is desired to compute electron transmission through the layer. Note that in Eyges (1949, 1950) and Blunck and Westphal (1951) the *path lengths* considered are small compared to the average range. This is not the case here, and electrons which lose most or all of their energy can be treated.

In the second paper, we delimit further the range of validity of the BRS transport equation, which defines all of its higher scattering moments as simple multiples of the first, instead of introducing many independent scattering moments from a carefully fitted elastic scattering cross section, or from experiment, as does the Spencer method. In that paper, the effect of the finite thickness of the layer will be introduced indirectly into the model for comparison of BRS and Spencer through an effective escape probability of bremsstrahlung photons from the layer. In the present paper, we consider the distribution of electrons in kinetic energy and path length, but no geometrical parameters such as finite layer thickness. Needless to say, the computer programs of Brown (1965), Brown and Ogilvie (1966), Ehrman and dePackh (1969, unpublished computer program 'Electrex') for electron transport and photon production in homogeneous layers of intermediate thickness treat boundary condition effects in detail. The present two papers serve to establish the conditions under which the electron transport differential equation program, with boundary conditions, is valid, and to give an estimate of the error involved in it. These error estimates will be discussed in detail in the second paper.

In § 2 of this paper, we estimate the straggling due to inelastic scattering, using the relativistic Møller scattering formula for free electrons. In § 3, we set up our model for the computation of radiative straggling, and show where it resembles and where it differs from that of Eyges. In § 4, we compute straggling parameters for certain bremsstrahlung spectra. In § 5, we relate our model to the computation of the differential range spectrum as a function of energy as done by Spencer and Fano. Section 6 summarizes the conclusions of the preceding sections.

2. Straggling due to inelastic scattering

For the continuous slowing down model of energy loss by moderately relativistic electrons in matter, we shall make use of the equation

$$-\frac{\mathrm{d}\gamma}{\mathrm{d}s} = 4\pi N Z r_0^2 \frac{\gamma^2}{\gamma^2-1} L_2 + \frac{\phi(\gamma, Z)}{137} N Z(Z+1)\gamma r_0^2 \tag{1}$$

where γ is the total energy of an electron in units of mc^2, s is the path length, N is the number of atoms per unit volume, r_0 is the so called classical electron radius e^2/mc^2, and L_2 and ϕ are defined below. The first term on the righthand side represents the loss of energy due to inelastic scattering by atomic electrons (both nonionizing and ionizing), while the second represents the energy loss due to bremsstrahlung. The 'logarithmic' factor L_2 is a slowly varying function of γ given by (Bethe and Ashkin 1953, p 254)

$$2L_2 = \ln\left\{\left(\frac{mc^2}{I}\right)^2 (\gamma-1)^2(\gamma+1)\right\} - 0{\cdot}568 - \frac{1{\cdot}636}{\gamma} + \frac{1{\cdot}818}{\gamma^2} \tag{2}$$

where I is an average excitation potential of the atom, approximately $(9{\cdot}5)Z$ eV. With this expression for L_2, $-\mathrm{d}\gamma/\mathrm{d}s$ of equation (1) represents the best value for average energy loss per unit length. The factor $\phi(\gamma, Z)$ in the second term of equation (1) is $\frac{16}{3}$ in the nonrelativistic Born approximation limit, and becomes

$$4\ln(183Z^{-1/3})+\tfrac{2}{9} \tag{3}$$

(see Heitler 1954, p 253) in the ultrarelativistic Born approximation limit. For $\gamma \leqslant 5$, the empirical formula

$$\phi = (6{\cdot}29+0{\cdot}0391Z)+(2{\cdot}17-0{\cdot}0170Z)\ln(\gamma-1)+(0{\cdot}661-0{\cdot}0104Z)\ln^2(\gamma-1) \tag{4}$$

fits experimental data on aluminium and gold (Koch and Motz 1959) fairly well. The factor $Z(Z+1)$ in equation (1) becomes Z^2 if bremsstrahlung due to atomic electrons is neglected. The neglect is an underestimate of bremsstrahlung, but replacing Z^2 by $Z(Z+1)$ is somewhat of an overestimate.

We are primarily concerned here with values of γ, Z low enough so that the second term on the righthand side of equation (1) is less than the first. The ratio of these two terms is subsequently called η. For the time being, we neglect the second term entirely, and determine the straggling due to inelastic scattering only. (The contribution of scattering of an electron by the atom as a whole to the straggling is neglected.)

Consider electrons of kinetic energy T, and define $P(T)$ as the derivative of the variance of energy loss with respect to path length for these electrons. Then (Evans 1955, p 661)

$$P(T) = \int_0^{T/2} Q^2 NZ\frac{\mathrm{d}\sigma}{\mathrm{d}Q}\,\mathrm{d}Q \tag{5}$$

where $\mathrm{d}\sigma(Q)/\mathrm{d}Q$ is the cross section for scattering of the incident electron by an atomic electron per unit range of energy loss Q. The upper limit on the integral in equation (5) is $\frac{1}{2}T$ rather than T, since the two electrons are identical particles, and the one emerging with the larger kinetic energy will be called the scattered electron. To compute the derivative of average energy loss with respect to path length, the factor Q^2 in equation (5) is replaced by Q. Then, the complicated behaviour of $\mathrm{d}\sigma/\mathrm{d}Q$ for Q so small that the binding of some or all of the atomic electrons is important must be taken into account. But the Q^2 in equation (5) makes this unnecessary (straggling is caused by a few large energy losses rather than by many small ones), and the Møller free electron expression may be used for $\mathrm{d}\sigma/\mathrm{d}Q$, that is (Evans 1955, p 577, equation (2.12))

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q} = \frac{2\pi e^4}{mv^2}\frac{1}{Q^2}\left(\frac{1}{1-q}\right)^2\left[1-\left\{3-\left(\frac{\gamma_0-1}{\gamma_0}\right)^2\right\}q(1-q)+\left(\frac{\gamma_0-1}{\gamma_0}\right)^2 q^2(1-q)^2\right]$$
$$\text{for } 0 \leqslant q \leqslant \tfrac{1}{2} \tag{6a}$$

where

$$q = \frac{Q}{T}. \tag{6b}$$

Carrying out the integration in equation (5) yields

$$P = 2\pi NZe^4\frac{\gamma^2}{\gamma+1}\left\{0{\cdot}4206+0{\cdot}2348\left(\frac{\gamma-1}{\gamma}\right)^2\right\}. \tag{7}$$

The variance in total path length or range for a given kinetic energy T_0

$$(\Delta(s^2))_{T_0} \equiv ((s^2)_{\mathrm{av}})_{T_0}-(s_{\mathrm{av}})^2_{T_0} \tag{8}$$

is given by

$$(\Delta(s^2))_{T_0} = \int_0^{T_0} \mathrm{d}T_{\mathrm{av}} \left| \frac{\mathrm{d}T_{\mathrm{av}}}{\mathrm{d}s} \right|^{-3} P(T_{\mathrm{av}})$$

$$= \frac{2\pi N Z e^4}{(mc^2)^2} \int_1^{\gamma} \mathrm{d}\gamma \left| \frac{\mathrm{d}\gamma_{\mathrm{av}}(s)}{\mathrm{d}s} \right|^{-3} \frac{\gamma^2}{\gamma+1} \left\{ 0{\cdot}4206 + 0{\cdot}2348 \left(\frac{\gamma-1}{\gamma} \right)^2 \right\}. \tag{9}$$

For $\mathrm{d}\gamma_{\mathrm{av}}/\mathrm{d}s$, insert the first term on the righthand side of equation (1), and assume $L_2(\gamma) = L_2(\gamma_0)$, since $L_2(\gamma)$ is an insensitive function of γ, especially if $\gamma - 1$ is not too small. Then

$$(\Delta(s^2))_{T_0} = \frac{J(\gamma_0)}{(2\pi N Z r_0^2)^2 8 L_2^3} \tag{10a}$$

where

$$J(\gamma) \equiv 0{\cdot}32769(\gamma^2 - 1) - \tfrac{9}{8}(\gamma - 1) - 0{\cdot}60631 \ln \gamma + 2{\cdot}01517\left(1 - \frac{1}{\gamma}\right) - 0{\cdot}37676\left(1 - \frac{1}{\gamma^2}\right)$$
$$- 0{\cdot}21846\left(1 - \frac{1}{\gamma^3}\right) + 0{\cdot}17611\left(1 - \frac{1}{\gamma^4}\right) - 0{\cdot}04696\left(1 - \frac{1}{\gamma^5}\right). \tag{10b}$$

Furthermore, using the same approximations as were used to derive equation (10), the range given by equation (1) is

$$s_{\mathrm{av}}(T_0) = \frac{\gamma_0 + (1/\gamma_0) - 2}{4\pi N Z r_0^2 L_2} \tag{11}$$

so that

$$\frac{(s^2)_{\mathrm{av}} - (s_{\mathrm{av}})^2}{(s_{\mathrm{av}})^2} = \frac{J(\gamma_0)}{\{\gamma_0 + (1/\gamma_0) - 2\}^2} \frac{1}{2L_2(\gamma_0)}. \tag{12}$$

For $\gamma_0 - 1 \ll 1$

$$J(\gamma_0) \simeq 0{\cdot}42056(\gamma_0 - 1)^4 \tag{13a}$$

while for $\gamma_0 - 1 \gg 1$

$$J(\gamma_0) \simeq 0{\cdot}32769\gamma_0^2. \tag{13b}$$

Under these conditions, equation (12) then yields

$$\frac{(s^2)_{\mathrm{av}} - (s_{\mathrm{av}})^2}{(s_{\mathrm{av}})^2} = \begin{cases} \dfrac{0{\cdot}210}{L_2} & \text{for } \gamma_0 - 1 \ll 1 \\ \dfrac{0{\cdot}164}{L_2} & \text{for } \gamma_0 - 1 \gg 1 \end{cases} \tag{14a}$$

or

$$\frac{\sqrt{(s^2)_{\mathrm{av}}}}{s_{\mathrm{av}}} - 1 = \begin{cases} \dfrac{0{\cdot}105}{L_2} & \text{for } \gamma_0 - 1 \ll 1 \\ \dfrac{0{\cdot}082}{L_2} & \text{for } \gamma_0 - 1 \gg 1. \end{cases} \tag{14b}$$

Table 1 gives some values of $L_2(\gamma)$ for various values of γ for aluminium ($Z = 13$) and uranium ($Z = 92$).

Table 1. Values of $L_2(\gamma)$ for various values of γ for Al and U

$\gamma - 1$	$L_2(\gamma)$, $Z = 13$	$L_2(\gamma)$, $Z = 92$
0·2	6·78	4·82
0·5	7·67	5·71
1	8·41	6·45
2	9·26	7·30
5	10·52	8·56
10	11·52	9·56
20	12·55	10·59
50	13·92	11·96
100	14·95	12·99

The results of table 1, together with equation (14), show that for $Z = 92$, $\gamma = 1{\cdot}2$ (ie primary kinetic energy about 100 keV), the value of $\sqrt{(s^2)_{\mathrm{av}}}/s_{\mathrm{av}}$ is not more than 1·0216, corresponding to a range straggling of slightly over 2 %, while for the other cases in table 1, it is even less. Since path length is the relevant parameter for photon production (except for a factor γ which varies little if $\gamma_0 = 1{\cdot}2$ is the primary γ), the effect of inelastic scattering of electrons on photon production, even when the primary energy is as low as 100 keV or even lower, is negligible.

3. Model for radiative straggling

In § 2, we have shown that for primary kinetic energies greater than or equal to 100 keV, the effect of straggling due to inelastic scattering on the range is of the order of 2 % or less. To estimate the magnitude of radiative straggling, therefore, we assume energy loss due to inelastic scattering to be a continuous slowing down process, as in the model of Eyges (1949). Furthermore, as in Eyges (1949), we shall assume the average energy loss rate due to inelastic scattering per unit path length, as well as the corresponding quantity for bremsstrahlung, and the shape of the bremsstrahlung spectrum (defined by $\hat{\Phi}(v)$ in equation (15) below), to be independent of electron kinetic energy. This assumption overestimates radiative straggling, since the effect of inelastic scattering increases and that of radiation decreases with decreasing electron energy. (The assumption can be removed by a somewhat more refined model not dealt with here.) We can start from equation (1) of Eyges (1949), where we assume all sources and fluxes homogeneous and isotropic, so that only energy and path length traversed from source enter the problem.

$$\frac{\partial \pi(\epsilon, s)}{\partial s} = -\int_0^1 \mathrm{d}v\hat{\Phi}(v)\left\{\pi(\epsilon, s) - \frac{1}{1-v}\pi\left(\frac{\epsilon}{1-v}, s\right)\right\} + \beta\frac{\partial \pi(\epsilon, s)}{\partial \epsilon} \tag{15}$$

where ϵ, s are normalized kinetic energy (ie kinetic energy/mc^2) and path length respectively (written as E, t by Eyges), β is normalized energy loss rate per unit length due to inelastic scattering, and $\hat{\Phi}(v)\,\mathrm{d}v\,\mathrm{d}s$ is the probability that in path length $\mathrm{d}s$, an electron emits a photon which has a fraction between v and $v + \mathrm{d}v$ of the electron kinetic energy. $\pi(\epsilon, s)\,\mathrm{d}\epsilon$ is proportional to the number of electrons with path length s traversed from their source and kinetic energy between ϵ and $\epsilon+\mathrm{d}\epsilon$ (p 269 of Rossi and Greisen 1941), and normalized so that

$$\int_0^\infty \mathrm{d}\epsilon\pi(\epsilon, 0) = 1. \tag{16}$$

Note that $\pi(\epsilon, s)$ is not the probability (as stated on p 264 of Eyges 1949) that an electron at path length s has energy between ϵ and $\epsilon+\mathrm{d}\epsilon$, since $\int_0^\infty \mathrm{d}\epsilon\pi(\epsilon, s) < 1$ for $s > 0$. (In fact, this is also pointed out on p 268 of Eyges 1949.)

If the source electrons are monochromatic of kinetic energy ϵ_0, then equation (15) is to be solved for $s > 0$ with the initial condition

$$\pi(\epsilon, 0) = \delta(\epsilon-\epsilon_0) \tag{17a}$$

and furthermore

$$\pi(\epsilon, s) = 0 \qquad \text{for all } \epsilon > \epsilon_0 \tag{17b}$$

and

$$\int_0^{\epsilon_0} \mathrm{d}\epsilon\pi(\epsilon, s) \tag{17c}$$

is the probability that the range of an emitted electron will be greater than or equal to s.

For $\hat{\Phi}(v)$, let us take

$$\hat{\Phi}(v) = G\left(\frac{\alpha}{v}+2(1-\alpha)\right) \tag{18}$$

where $0 \leqslant \alpha \leqslant 2$. The lower the value of α, the higher is the contribution of the harder photons to the total photon energy production, and the larger is the radiative straggling. The ratio

$$\frac{\int_{1/2}^1 v\hat{\Phi}(v)\,\mathrm{d}v}{\int_0^{1/2} v\hat{\Phi}(v)\,\mathrm{d}v} = \frac{3-\alpha}{1+\alpha} \tag{19}$$

may be taken as a measure of the contribution to the energy output of the hard half of the spectrum compared to the soft, and has the values 3, 1, $\frac{1}{3}$ for $\alpha = 0, 1, 2$ respectively. The case $\alpha = 1$ corresponds to a bremsstrahlung spectrum of constant photon intensity per unit frequency, and will be dealt with in more detail than $\alpha \neq 1$. A somewhat softer spectrum (α somewhat greater than 1) is closer to experimental results. Define

$$x = \frac{\beta}{\epsilon_0}s \tag{20a}$$

$$\tau = \frac{\epsilon}{\epsilon_0} \tag{20b}$$

$$\sigma = \frac{\epsilon}{\epsilon_0(1-x)} \tag{20c}$$

$$\eta = \frac{G\epsilon_0}{\beta}. \tag{20d}$$

x is a dimensionless path length, defined so that in the absence of bremsstrahlung loss, each electron would run from $x = 0$ to $x = 1$. Because of the radiative losses, different particles have different ranges, corresponding to x values less than 1. σ is the ratio of the kinetic energy which an electron of path length x has to what it would have if it had suffered no loss by bremsstrahlung. Clearly $0 \leqslant \sigma \leqslant 1$. Note that η has the physical significance of the ratio of average energy loss rate by radiation of a source electron to the energy loss rate by inelastic scattering, since $\int_0^1 v\hat{\Phi}(v)\,dv = G$ by equation (18). Using the ratio of the second to the first term on the righthand side of equation (1) for η, with $\phi(\gamma, Z)$ estimated from the Born approximation values given by Heitler (1954, pp 251–3) table 2 of approximate η values is obtained

Table 2. Values of η for various values of γ for Al and U

$\gamma-1$	$\eta(Z=13)$	$\eta(Z=92)$
0·2	0·00236	0·0220
0·5	0·00478	0·0427
1	0·0080	0·069
2	0·0152	0·128
5	0·0411	0·322
10	0·086	0·63
20	0·175	1·22
50	0·435	2·90
100	0·86	5·58

Unlike Eyges (1949), we do not begin with the approximation $\beta = 0$ (ie $\eta = \infty$) and then attempt to expand the relevant functions in a power series in s for finite β. In fact, it turns out that $s = 0$ (or $x = 0$) is a logarithmic branch point of these functions (see equation (43a, b) below) so that an expansion in powers of the path length is not possible. (This explains the infinite coefficient of t^3 in $M(0, t)$ on p 268 of Eyges (1949).) Instead, we assume the non-negative variable to be small enough so that a series in powers of η is useful with a small number of terms. It turns out that for $\eta \leqslant \frac{1}{2}$, this is certainly the case. This will be made quantitative in § 4. The coefficients of the first few powers of η will be functions of x, some of them with a logarithmic branch point at $x = 0$ (see equation (43a, b) below).

Define the dimensionless quantity

$$\hat{\pi}(\sigma, x) = (1-x)\epsilon_0 \pi(\epsilon, s) \tag{21}$$

and its Mellin transform

$$g(\rho, x) = \int_0^\infty d\sigma \sigma^{\rho-1} \hat{\pi}(\sigma, x). \tag{22}$$

It is more convenient to work with σ than with ϵ or τ since, in the absence of bremsstrahlung ($\eta = 0$)

$$\hat{\pi}(\sigma, x) = \delta(1-\sigma) \tag{23}$$

for all x, not merely for $x = 0$. In general, $\hat{\pi}(\sigma, x)$ satisfies

$$\frac{\partial \hat{\pi}(\sigma, x)}{\partial x} + \frac{\hat{\pi}(\sigma, x)}{1-x} - \frac{1-\sigma}{1-x}\frac{\partial \hat{\pi}(\sigma, x)}{\partial \sigma}$$
$$= -\eta \int_0^1 dv \left(\frac{\alpha}{v} + 2(1-\alpha)\right)\left(\hat{\pi}(\sigma, x) - \frac{1}{1-v}\hat{\pi}\left(\frac{\sigma}{1-v}, x\right)\right) \tag{24a}$$

with

$$\hat{\pi}(\sigma, 0) = \delta(1-\sigma) \tag{24b}$$

and the Mellin transform $g(\rho, x)$ satisfies

$$\frac{\partial g(\rho, x)}{\partial x} + \frac{1-\rho}{1-x}(g(\rho, x) - g(\rho-1, x)) = -\eta\left(\alpha\Phi(\rho-1) + 2(1-\alpha)\frac{\rho-1}{\rho}\right)g(\rho, x) \tag{25a}$$

with

$$g(\rho, 0) = 1. \tag{25b}$$

In equation (25*a*)

$$\Phi(\rho) = \Psi(\rho) + C = \sum_{\nu=1}^{\infty}\left(\frac{1}{\nu} - \frac{1}{\rho+\nu}\right) \tag{26}$$

where $\Psi(\rho)$ is the logarithmic derivative of the factorial function (Jahnke and Emde 1938, p 18) and C is Euler's constant, so that $\Phi(0) = 0$. $\Phi(\rho)$ has a pole for each real negative integer. Examining equation (25*a*) suggests that it may be useful to define two auxiliary functions, $T(\rho, x)$ and its inverse Mellin transform $t(\sigma, x)$, by

$$T(\rho, x) = g(\rho, x)(1-x)^{\rho-1}\exp\left\{\eta x\left(\alpha\Phi(\rho-1) + 2(1-\alpha)\frac{\rho-1}{\rho}\right)\right\} \tag{27}$$

$$t(\sigma, x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} T(\rho, x)\sigma^{-\rho}\,d\rho \tag{28}$$

where c can be any positive real number, since $\int_0^\infty d\sigma|\sigma^{c-1}t(\sigma, x)|$ converges for all real positive c. (Note $t(\sigma, x) = 0$ for $\sigma > 1$.) Then $T(\rho, x)$ satisfies the equation

$$\frac{\partial T(\rho, x)}{\partial x} = (1-\rho)\exp\left\{\eta x\left(\frac{\alpha}{\rho-1} + \frac{2(1-\alpha)}{\rho(\rho-1)}\right)\right\}T(\rho-1, x) \tag{29a}$$

with

$$T(\rho, 0) = 1. \tag{29b}$$

Equation (29*a*) has an important advantage over equation (25*a*). The exponential on the righthand side of equation (29*a*) has singularities only for $\rho = 0$ and $\rho = 1$, while on the righthand side of equation (25*a*) are singularities for every real nonpositive integer. For $\eta = 0$, both equations are easy to solve, with the exact solutions

$$g(\rho, x) = 1 \qquad \text{for } x < 1 \tag{30a}$$

$$T(\rho, x) = (1-x)^{\rho-1}. \tag{30b}$$

(Of course, $g(\rho, x) = 0$ for $x > 1$.) We shall now solve equation (29*a*) for η positive but small, say less than or equal to $\frac{1}{2}$. (How large η may be in practice is discussed in § 4.) For this purpose, we note that always $0 \leqslant x \leqslant 1$ and $0 \leqslant \alpha \leqslant 2$, so that it is legitimate to expand the exponential in equation (29*a*) in a power series in η, provided that ρ is not close to 0 or 1. We can thus guess a solution for $T(\rho, x)$ close to the solution equation (30*b*) for $\text{Re}(\rho-1)$ positive and not too close to zero. In practice, it will be more convenient to work with a differential equation for $t(\sigma, x)$ than with a difference equation for $T(\rho, x)$, as indicated below.

The behaviour of the Mellin transform is such that if the function $\hat{\Phi}(v)$ of equation (18) were replaced by a slightly different function, say by

$$\tilde{\Phi}(v\,;v_0) = \begin{cases} \dfrac{1}{v_0}\hat{\Phi}(v) & \text{for } 0 < v < v_0 \\ 0 & \text{for } v_0 < v < 1 \end{cases} \tag{31}$$

then in the half plane $\operatorname{Re}\rho < 0$, the resulting $T(\rho, x)$ for v_0 close to 1 (eg 0·999) would be a very different function from our $T(\rho, x)$ resulting from $\hat{\Phi}(v) \equiv \tilde{\Phi}(v\,;1)$. But this instability of $T(\rho, x)$ as a functional of $\tilde{\Phi}(v\,;v_0)$ does not appear in the $\operatorname{Re}\rho > 0$ half plane where we need $T(\rho, x)$. Hence, equation (29) may be solved for $T(\rho, x)$ approximately for η positive but not too large in the half plane $\operatorname{Re}\rho > 0$. To compute the average value of the nth power of the kinetic energy for fixed x, we require $g(1, x)$ and $g(n+1, x)$ (and hence $T(1, x)$ and $T(n+1, x)$) for that x. To compute range straggling, we require $g(1, x)$ (and hence $T(1, x)$) as a function of x.

In equation (29*a*), assume $\operatorname{Re}(\rho-1) > 0$ and $|\rho-1|$, $|\rho(\rho-1)|$ sufficiently large so that the Taylor series of the exponential converges rapidly (to be made quantitative in § 4). First, consider $\alpha = 1$ with only the 0th and 1st terms in the Taylor series; then consider $\alpha = 1$ with the 0th, 1st, and 2nd terms, and $\alpha \neq 1$ with the 0th and 1st terms.

With $\alpha = 1$ and only the 0th and 1st terms in the Taylor series for the exponential in equation (29a) retained, we get

$$\frac{\partial T(\rho, x)}{\partial x} = (1-\rho-\eta x)T(\rho-1, x) \tag{32}$$

with

$$T(\rho, 0) = 1.$$

This gives for the inverse Mellin transform $t(\sigma, x)$

$$\frac{\partial t(\sigma, x)}{\partial x} = \frac{\partial t(\sigma, x)}{\partial \sigma} - \frac{\eta x}{\sigma} t(\sigma, x) \tag{33a}$$

with

$$t(\sigma, 0) = \delta(1-\sigma). \tag{33b}$$

Equations (33) can easily be solved to give

$$t(\sigma, x) = e^{\eta x}(1-x)^{\eta}\delta(1-\sigma-x) \tag{34}$$

whence

$$T(\rho, x) = e^{\eta x}(1-x)^{\eta+\rho-1} \tag{35}$$

which of course reduces to equation (30*b*) for $\eta = 0$.

If an extra term (the η^2 term) is retained in the exponential of equation (29*a*) and if α is again set equal to 1, then

$$\frac{\partial T(\rho, x)}{\partial x} = \left(1-\rho-\eta x - \frac{\eta^2 x^2}{2(\rho-1)}\right) T(\rho-1, x). \tag{36}$$

This gives for the inverse Mellin transform $t(\sigma, x)$

$$\frac{\partial t(\sigma, x)}{\partial x} = \frac{\partial t(\sigma, x)}{\partial \sigma} - \frac{\eta x}{\sigma} t(\sigma, x) - \frac{\eta^2 x^2}{2\sigma} \int_\sigma^\infty d\sigma' \frac{t(\sigma', x)}{\sigma'}. \tag{37}$$

To solve equation (37), use $\xi = \sigma + x$ and x in place of σ and x as independent variables. Then

$$\frac{\partial t(\xi, x)}{\partial x} + \frac{\eta x}{\xi - x} t = -\frac{\eta^2 x^2}{2(\xi - x)} \int_\xi^\infty d\xi' \frac{t(\xi', x)}{\xi' - x} \tag{38}$$

whence

$$\frac{\partial^2 t}{\partial x\, \partial \xi} = -\frac{1}{\xi - x}\frac{\partial t}{\partial x} - \frac{\eta x}{\xi - x}\frac{\partial t}{\partial \xi} + \frac{\eta^2 x^2}{2(\xi - x)^2} t \tag{39}$$

with the boundary conditions

$$t(\xi, 0) = \delta(\xi - 1) \tag{40a}$$

$$t(M, x) = 0 \qquad \text{for any } M > 1. \tag{40b}$$

This hyperbolic differential equation may be solved iteratively as a Goursat's problem (Garabedian 1964, p 117). Consistently neglecting terms of order η^3, it turns out that

$$t(\xi, x) = \delta(\xi - 1)\, e^{\eta x}(1 - x)^\eta - \frac{\eta^2}{2} U(1 - \xi) \int_0^x dx' \frac{x'^2(1 - x')^{\eta - 1}}{\xi - x'} \tag{41a}$$

where $U(x) = 0$ for $x < 0$ and 1 for $x > 0$. Alternatively

$$t(\sigma, x) = \delta(\sigma + x - 1)\, e^{\eta x}(1 - x)^\eta - \frac{\eta^2}{2} U(1 - \sigma - x) \int_0^x dx' \frac{x'^2(1 - x')^{\eta - 1}}{\sigma + x - x'}. \tag{41b}$$

Hence, from equation (41b)

$$T(\rho, x) = e^{\eta x}(1 - x)^{\eta + \rho - 1} - \frac{\eta^2}{2} \int_0^x dx' x'^2 (1 - x')^{\eta - 1} \int_0^{1 - x} d\sigma \frac{\sigma^{\rho - 1}}{\sigma + x - x'} \tag{42}$$

where $e^{\eta x}$ may be replaced by $1 + \eta x + \frac{1}{2}\eta^2 x^2$, since terms of order η^3 are neglected. In particular

$$g(1, x) = T(1, x) = e^{\eta x}(1 - x)^\eta - \frac{\eta^2}{2} \int_0^x dx' x'^2 (1 - x')^{\eta - 1} \ln\left(\frac{1 - x'}{x - x'}\right) \tag{43a}$$

which gives, if η is put equal to 0 inside the integral

$$\begin{aligned} g(1, x) &= T(1, x) \\ &= e^{\eta x}(1 - x)^\eta - \tfrac{1}{2}\eta^2[(x + \tfrac{1}{2}x^2)\ln x + G(x) + \tfrac{1}{4}(1 - x)\{x + (3 + x)\ln(1 - x)\}] \end{aligned} \tag{43b}$$

where

$$G(x) = -\int_0^x dx' \frac{\ln(x')}{1 - x'}. \tag{43c}$$

Note that $G(0) = 0$ and $G(1) = \frac{1}{6}\pi^2$ and that $G(x)$ is a monotonic increasing function of x on $0 \leqslant x \leqslant 1$. As stated above, the coefficient of η^2 in $g(1, x)$ has a branch point at $x = 0$. However, $g(1, x)$ in equation (43b) does not go to zero as $x \to 1$. This is because putting $\eta = 0$ inside the integral of equation (43a) is not legitimate for x near 1. If x is near 1, then, neglecting $O(\eta^3)$

$$g(1, x) = e^{\eta x}(1 - x)^\eta - \frac{\pi^2}{12}\eta^2(1 - x)^\eta(1 + R) \tag{44}$$

where the remainder term R goes to zero at least as fast as $(1 - x)^{1 - \eta}$ as $x \to 1$.

Going back now to equation (29a) for arbitrary α on (0, 2), and taking only the 0th and 1st terms in the Taylor series of the exponential, we get

$$\frac{\partial T(\rho, x)}{\partial x} = \left(1-\rho-\alpha\eta x-\frac{2(1-\alpha)}{\rho}\right) T(\rho-1, x) \tag{45}$$

instead of equation (36). Instead of equation (37), we get

$$\frac{\partial t(\sigma, x)}{\partial x} = \frac{\partial t(\sigma, x)}{\partial \sigma}-\frac{\alpha\eta x}{\sigma} t(\sigma, x)-2(1-\alpha)\eta x \int_{\sigma}^{\infty} \frac{d\sigma'}{\sigma'^2} t(\sigma', x). \tag{46}$$

Again, use $\xi = \sigma + x$ and x as independent variables then

$$\frac{\partial t(\xi, x)}{\partial x}+\frac{\alpha\eta x}{\xi-x} t(\xi, x) = -2(1-\alpha)\eta x \int_{\xi}^{\infty} d\xi' \frac{t(\xi', x)}{(\xi'-x)^2}. \tag{47}$$

Inspecting equation (41a), we are led to the *ansatz*

$$t(\xi, x) = \delta(\xi-1)\, e^{\eta x}(1-x)^{\eta}+U(1-\xi) G(\xi, x) \tag{48a}$$

with the boundary condition

$$G(\xi, 0) = 0. \tag{48b}$$

Inserting (48a) into (47) gives, for $0 \leqslant x < \xi < 1$

$$\frac{\partial G(\xi, x)}{\partial x}+\frac{\alpha\eta x}{\xi-x} G(\xi, x) = -2(1-\alpha)\eta x \frac{e^{\alpha\eta x}(1-x)^{\alpha\eta}}{(1-x)^2}-2(1-\alpha)\eta x \int_{\xi}^{1} d\xi' \frac{G(\xi', x)}{(\xi'-x)^2}. \tag{49}$$

Putting $\xi = 1$ in equation (49) we may solve the resulting ordinary differential equation for $G(1, x)$, using (48b) for a boundary condition at $x = 0$, obtaining

$$G(1, x) = -2(1-\alpha)\eta\, e^{\alpha\eta x}(1-x)^{\alpha\eta}\left(\frac{x}{1-x}+\ln(1-x)\right). \tag{50}$$

Taking $\partial/\partial\xi$ of equation (49) yields

$$\frac{\partial^2 G}{\partial\xi\, \partial x} = (2-\alpha)\eta x \frac{G(\xi, x)}{(\xi-x)^2}-\frac{\alpha\eta x}{\xi-x} \frac{\partial G(\xi, x)}{\partial \xi} \tag{51}$$

which, together with the boundary conditions (48b) and (50), can be solved as a Goursat problem. Consistently neglecting terms of order η^2, we obtain

$$t(\xi, x) = \delta(\xi-1)\, e^{\alpha\eta x}(1-x)^{\alpha\eta}-2(1-\alpha)\eta\, U(1-\xi)\, e^{\alpha\eta x}(1-x)^{\alpha\eta}\left(\frac{x}{1-x}+\ln(1-x)\right) \tag{52a}$$

or

$$t(\sigma, x) = \delta(\sigma+x-1)\, e^{\alpha\eta x}(1-x)^{\alpha\eta}-2(1-\alpha)\eta\, U(1-\sigma-x)\, e^{\alpha\eta x}(1-x)^{\alpha\eta}\left(\frac{x}{1-x}+\ln(1-x)\right) \tag{52b}$$

whence

$$T(\rho, x) = e^{\alpha\eta x}(1-x)^{\alpha\eta+\rho-1}\left(1-\frac{2(1-\alpha)\eta}{\rho}\{x+(1-x)\ln(1-x)\}\right). \tag{53}$$

Since we neglect terms of order η^2, $e^{\alpha\eta x}$ in equation (53) may be replaced by $1+\alpha\eta x$.

In particular

$$g(1,x) = T(1,x) = e^{\alpha\eta x}(1-x)^{\alpha\eta}[1-2(1-\alpha)\eta\{x+(1-x)\ln(1-x)\}]. \tag{54}$$

4. Results for radiative straggling

The results of the previous section may be used to compute the moments of the kinetic energy for fixed path length x, and more important, to compute the range straggling. For these purposes, it is not necessary to invert $g(\rho, x)$ and obtain $\hat{\pi}(\sigma, x)$. Instead, one can work directly with $g(\rho, x)$ for suitable ρ.

To compute range straggling, note that since, by equation (22), $g(1,x) = \int_0^\infty d\sigma\hat{\pi}(\sigma, x)$, it follows that:

$$x_{av} = \int_0^1 dx x\left(-\frac{dg(1,x)}{dx}\right) \tag{55a}$$

and

$$(x^2)_{av} = \int_0^1 dx x^2\left(-\frac{dg(1,x)}{dx}\right). \tag{55b}$$

Using integration by parts and noting that $g(1, 1) = 0$

$$x_{av} = \int_0^1 g(1,x)\,dx \tag{56a}$$

$$(x^2)_{av} = 2\int_0^1 xg(1,x)\,dx. \tag{56b}$$

Using equation (54) for $g(1, x)$ and neglecting $O(\eta^2)$

$$x_{av} = 1-\tfrac{1}{2}\eta \tag{57a}$$

$$(x^2)_{av} = 1-(\tfrac{7}{9}+\tfrac{1}{18}\alpha)\eta \tag{57b}$$

and

$$\frac{(x^2)_{av}-(x_{av})^2}{(x_{av})^2} = \frac{\eta}{18}(4-\alpha) \tag{57c}$$

$$\frac{\sqrt{(x^2)_{av}}}{x_{av}}-1 = \frac{\eta}{36}(4-\alpha). \tag{57d}$$

The quantity in equation (57*d*) may be taken as a measure of the range straggling. For $\alpha = 0$, it is $\eta/9$, for $\alpha = 1$, it is $\eta/12$, and for $\alpha = 2$, it is $\eta/18$. As expected, the softer the spectrum, that is, the higher the value of α, $0 \leqslant \alpha \leqslant 2$, the less straggling there is.

If it is desired to compute the average value of σ and σ^2 for fixed x, note that

$$\langle\sigma^n\rangle = \frac{g(n+1,x)}{g(1,x)}. \tag{58}$$

Using equations (53) and (27) and neglecting $O(\eta^2)$

$$\sigma_{av} = e^{-\eta x}[1+(1-\alpha)\eta\{x+(1-x)\ln(1-x)\}] \tag{59a}$$

$$(\sigma^2)_{av} = \exp\{-(\tfrac{1}{6}\alpha+\tfrac{4}{3})\eta x\}[1-\tfrac{4}{3}(1-\alpha)\eta\{x+(1-x)\ln(1-x)\}] \tag{59b}$$

$$\frac{(\sigma^2)_{\text{av}}}{(\sigma_{\text{av}})^2} = \exp\{+(\tfrac{2}{3}-\tfrac{1}{6}\alpha)\eta x\}[1-\tfrac{2}{3}(1-\alpha)\eta\{x+(1-x)\ln(1-x)\}]$$
$$= 1+\eta x(\tfrac{2}{3}-\tfrac{1}{6}\alpha)-\tfrac{2}{3}(1-\alpha)\eta\{x+(1-x)\ln(1-x)\}. \tag{59c}$$

For $\alpha = 1$, use equation (43*a*) and equation (56) to compute range straggling, neglecting terms of order η^3. Then

$$x_{\text{av}} = 1-\tfrac{1}{2}\eta+\tfrac{1}{4}\eta^2 \tag{60a}$$

$$(x^2)_{\text{av}} = 1-\tfrac{5}{6}\eta+\tfrac{73}{144}\eta^2 \tag{60b}$$

$$\frac{(x^2)_{\text{av}}-(x_{\text{av}})^2}{(x_{\text{av}})^2} = \frac{\eta}{6}\left(1-\frac{11}{24}\eta\right) \tag{60c}$$

$$\frac{\sqrt{(x^2)_{\text{av}}}}{x_{\text{av}}}-1 = \frac{\eta}{12}\left(1-\frac{1}{2}\eta\right). \tag{60d}$$

The errors involved in the approximations leading to equations (57*d*) and (60*d*) have two sources. One is the replacement of the exponential in equation (29*a*) by a power series. The worst error is that for $x = 1$ and $\rho = 2$, which means e^{η} is replaced by $1+\eta$ in equations (57) and by $1+\eta+\frac{1}{2}\eta^2$ in equations (60). For $\eta = \frac{1}{2}$, these errors are only 9% and $1\frac{1}{2}$% respectively, and even for $\eta = 1$, they are 26% and 8% respectively. Much more serious is the replacement of terms like $1/(1+\eta)$ by $1-\eta$ or $1-\eta+\eta^2$ in subsequent steps where terms of order η^2 or η^3 have been systematically dropped *after* the differential equations (39) and (51) had been set up. These approximations make it impossible to use equations (57) and (60) up to $\eta = 1$, and indicate that the solutions given here should not be relied upon for $\eta > \frac{1}{2}$. Equation (60*d*) indicates that for $\alpha = 1$, equation (57*d*) is subject to a relative error of about $\frac{1}{2}\eta$. When computing range (x_{av}) in equation (57*a*) for $\alpha = 1$, we see by equation (60*a*) that the relative error in range by neglecting $O(\eta^2)$ is only about $\frac{1}{4}\eta^2$. Since the error in going from equation (29*a*) to equation (39) or (51) is less than the subsequent error in going from equation (39) to equation (60*a*) from equation (51) to equation (57), it is *consistent* to keep a higher degree in η in solving equation (39) or (51) and in the subsequent steps than has been retained in deriving equation (39) or (51). An improvement in equations (60) and (57) can thus be obtained without too much trouble.

Table 2 shows that the present restriction $\eta \leqslant \frac{1}{2}$ means an electron kinetic energy less than or equal to 4 MeV in uranium and less than or equal to 30 MeV in aluminium.

It may be argued that, because of the factor γ in the second term on the righthand side of equation (1), the relative error in photon energy production due to neglect of straggling is better approximated by $(\sqrt{(y^2)_{\text{av}}}/y_{\text{av}})-1$ than by $(\sqrt{(x^2)_{\text{av}}}/x_{\text{av}})-1$, where

$$\frac{\mathrm{d}y(x)}{\mathrm{d}x} = \frac{\langle\gamma(x)\rangle}{\gamma_0} \simeq \frac{\langle\epsilon(x)\rangle}{\epsilon_0} = (1-x)\langle\sigma(x)\rangle \tag{61}$$

assuming $\gamma \gg 1$ and $\phi(\gamma, Z)$ independent of γ. For the case $\alpha = 1$, with $O(\eta^2)$ neglected, we obtain

$$\frac{\sqrt{(y^2)_{\text{av}}}}{y_{\text{av}}}-1 = \frac{\eta}{40} \tag{62}$$

as compared with $\eta/12$ from equation (57*d*) or (60*d*). In deriving equation (62), it is important not to replace $e^{-\eta x}$ by $1-\eta x$ for σ_{av} prematurely, since the terms for $(y^2)_{\text{av}}$

which would then be changed are not all $O(\eta^2)$. If we assume that both inelastic scattering and radiation energy loss rates per unit length are independent of electron energy, then range straggling (equations (57*d*), (60*d*)) is a measure of the error in photon production due to straggling neglect. If we assume that both are proportional to electron kinetic energy, then equation (62) is a measure of this error. In fact, the second assumption is good for radiative loss (if $\gamma \gg 1$), but inelastic scattering loss changes little with decreasing electron energy as long as $\gamma \gg 1$ (see equation (1)). This means that the effective η is decreasing as electron energy decreases. Since we use η_{primary} for our constant η, we do not underestimate the error in total photon production energy in a continuous slowing down model.

5. Relationship of the model to the Spencer–Fano electron energy spectrum

As a check on the model of § 3, it can be shown to yield the same qualitative behaviour for the electron-slowing down spectrum as a function of kinetic energy as the more detailed calculation of that function by Spencer and Fano (1954). Let $y(T)$ (where $T = mc^2\epsilon$) represent this slowing down function or differential track length (dimension: length energy^{-1}); $S\delta(T_0 - T)$ the source strength (dimension: energy^{-1}), S being total number of electrons emitted and therefore dimensionless; and $k(T, \tau)\,\mathrm{d}\tau$ the probability per unit path length that an electron of energy T experiences an energy loss between τ and $\tau + \mathrm{d}\tau$. Then, in our model

$$k(T, \tau) = -\beta\delta'(T) + \frac{\eta\beta}{T}\left(\frac{\alpha}{\tau} + 2(1-\alpha)\right) \tag{63}$$

where the first term represents the continuous slowing-down by inelastic scattering and the second represents radiative energy loss. Equations (13) and (14) of Spencer and Fano become for a monochromatic source

$$K(T', T) = \int_{T'-T}^{\infty} K(T', \tau)\,\mathrm{d}\tau \tag{64a}$$

$$\int_T^{\infty} \mathrm{d}T' y(T') K(T', T) = SU(T_0 - T). \tag{64b}$$

If we expand

$$y(T) = y_0(T) + \eta y_1(T) + \ldots \tag{65}$$

we obtain at once, with the assumption of an energy independent β

$$y_0(T) = \frac{S}{\beta} U(T_0 - T) \tag{66}$$

and, for $\alpha = 1$, by iteration in powers of η

$$y_1(T) = -\frac{S}{\beta} G\left(1 - \frac{T}{T_0}\right) U(T_0 - T) \tag{67}$$

where $G(x)$ is defined by equation (43*c*). Note that for $T < T_0$

$$\frac{\mathrm{d}y_1(T)}{\mathrm{d}T} = +\frac{S}{\beta T} \ln\left(\frac{T_0}{T_0 - T}\right) \tag{68}$$

so that

$$\lim_{T \to T_0^-} \frac{\mathrm{d}y_1(T)}{\mathrm{d}T} = +\infty$$

in accordance with the graphs of curves III in figures 1, 2 and 3 of Spencer and Fano (1954). Note that no comparison with curve III should be made for T appreciably less than $\frac{1}{2}T_0$, since the production of secondary electrons is neglected in our model. The existence of a marked minimum in curve III in figure 1 of Spencer and Fano ($y(T)$ for $T_0 = 80mc^2$, $Z = 82$) is not reproduced by our equations (65), (66) and (67) since it is a high η effect. However, the infinite slope of $y_1(T)$ as a function of T for $T \to T_0$ from below already follows from the coefficient of the first power η in the series of equation (65).

6. Conclusions

It has been shown that the neglect of electron straggling, particularly in a computation for bremsstrahlung photon production, is a much better assumption than has generally been believed, under a fairly wide range of parameters, even for moderately relativistic energies. Range straggling due to inelastic scattering is only slightly over 2% for 100 keV electrons in uranium, and goes down as energy goes up or as Z goes down. Range straggling due to bremsstrahlung losses was therefore computed considering inelastic scattering as a continuous slowing down process. While it is trivial that as η (equation (20*d*)) goes to zero, radiative range straggling must go to zero, it is a very significant result that the range straggling is only $\eta/12$ in a model which, in fact, overestimates radiative straggling by taking β and η energy independent (at their primary values) rather than considering their energy dependence. Equation (1) may be used to give their approximate energy dependence, and an expansion in a power series of η_{primary} can then be made, which gives a refinement of the present model just as the second Eyges paper on straggling (Eyges 1950) provides a refinement of the first (Eyges 1949). Again, no Taylor series expansion in x or path length s about zero is to be made. Furthermore, the spectrum of equation (18) can be generalized to

$$\hat{\Phi}(\epsilon, v) = G\left\{\frac{\alpha}{v} + 2\left(1 - \alpha - \frac{A}{3}\right) + Av\right\} \tag{69}$$

where G, α, A are all suitable functions of ϵ, and α, A give the photon spectrum shape for fixed ϵ. Again, one need only consider the Mellin transforms $g(\rho, x)$ for $\rho = 1$, and also for $\rho = 2, 3$ if $\langle\sigma\rangle$, $\langle\sigma^2\rangle$ are desired for fixed x, rather than the more complicated $\hat{\pi}(\sigma, x)$.

The derivation of equation (62) shows that the error in photon production due to straggling neglect is less than the straggling in electron range. The maximum η to which equations (57) and (60) are valid is given by the error involved in replacing e^{η} by $1+\eta$ or $1+\eta+\frac{1}{2}\eta^2$ (see equation (29*a*) with $\rho = 2$, $x = 1$), as well as the errors in subsequent steps in the derivation of equations (57) and (60) where, for example, $1/(1+\eta)$ was replaced by $1-\eta$ or $1-\eta+\eta^2$. The latter step places a lower bound on permissible η (or a higher error for fixed η) than that in equation (29*a*). For $\eta = \frac{1}{2}$, the range straggling is given by $\eta/12$ or 4% to within a good approximation. However, if the range straggling is desired for $\eta = 1$, then the transition from equation (29*a*) to equation (36) is still satisfactory within a 10% error limit, but one must be careful not to expand $1/(1+\eta)$ in a Taylor series. It is not difficult to replace equation (60*d*) by an equation which is good within a 10% error limit for our model up to $\eta = 1$.

We have set up a model which can only overestimate range straggling, and which gives 3 % range straggling for a fairly hard spectrum $\alpha = 1$ for $\eta = \frac{1}{2}$ (equation (60*d*) with $\eta = \frac{1}{2}$). The figure of 3 % is subject to relative error conservatively estimated at η^2 (ie 25 %) within the framework of the overestimating model.

Hence, for $\eta \leqslant \frac{1}{2}$, that is, primary electron energies less than or equal to 4 MeV in uranium or less than or equal to 30 MeV in aluminium, bremsstrahlung may be considered as a continuous slowing down process, with an error of at most 4 % in range straggling and in photon energy production. This justifies the use of a continuous slowing down treatment for electron transport in computations of photon production by Brown (1965), Brown *et al* (1969), and by Ehrman and dePackh (1969, unpublished computer program 'Electrex').

It should be noted that for a primary electron kinetic energy as low as 100 keV, the production of photons by K fluorescence from discrete levels is not negligible compared to bremsstrahlung. For low Z, direct fluorescence is more important than indirect fluorescence; for high Z, the reverse is the case. Also, as shown in § 2, for primary energies below 100 keV, range straggling due to inelastic scattering rises above 2 %.

Acknowledgments

The author wishes to thank Professor A S Deakin for an interesting discussion on integral transforms. This research was carried out under a grant from the National Research Council of Canada, which is gratefully acknowledged.

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